J. Math. Anal. Appl. 445 (2017) 337–341

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Spherical isometries of finite dimensional C^*-algebras

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ARTICLE INFO

Article history:
Received 14 January 2016
Available online 4 August 2016
Submitted by M. Mathieu

Keywords:
Isometries
Unit sphere
Unitary group
Faces
Jordan isomorphisms

ABSTRACT

In this paper, it is shown that every surjective isometry between the unit spheres of two finite dimensional C^*-algebras extends to a real-linear Jordan $*$-isomorphism followed by multiplication operator by a fixed unitary element. This gives an affirmative answer to Tingley's problem between two finite-dimensional C^*-algebras. Moreover, we show that if two finite dimensional C^*-algebras have isometric unit spheres, then they are $*$-isomorphic.

1. Introduction

Throughout this paper, all C^*-algebras are assumed to be unital. For a Banach space X, let $B(X)$ and $S(X)$ be the unit ball and unit sphere of X, respectively. This paper is concerned with the following problem.

Tingley's problem *(Tingley [21] 1987).* Let X and Y be Banach spaces. Suppose that $T_0 : S(X) \to S(Y)$ is a surjective isometry. Does there exist a real-linear isometric isomorphism $T : X \to Y$ satisfying $T|S(X) = T_0$.

The origin of Tingley's problem is the celebrated Mazur–Ulam theorem which states that every surjective isometry between normed spaces is automatically affine. This means, in a sense, that the (real) algebraic structure of a normed space is determined by its metric structure. Furthermore, in 1972, Mankiewicz [15] generalized the Mazur–Ulam theorem by showing that every surjective isometry between open connected subsets of real normed spaces is uniquely extended to an affine isometry between the whole spaces. In particular, a surjective isometry between the unit balls of two normed spaces extends to a real-linear isometric isomorphism. Inspired from this, Tingley [21] considered surjective isometries between unit spheres of normed space; and then asked whether or not they extend to real-linear isometries. Many papers have been devoted to studying Tingley's problem; see, e.g., [5,6,10] for recent development on the problem. However,

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http://dx.doi.org/10.1016/j.jmaa.2016.07.073

Tingley's problem is still open even in the two-dimensional case. The survey of Ding [4] is one of the good starting points for understanding the problem.

There is another setting of Mazur–Ulam type problem important in this paper, that is, the case of unitary groups of C^*-algebras. In [8], Hatori and Molnár completely determined the forms of surjective isometries on the unitary group of the algebra of all bounded linear operators $\mathscr{B}(\mathscr{H})$ on a complex Hilbert space $\mathscr{H}$. From this result, in particular, it turned out that every surjective isometry on the unitary group of $\mathscr{B}(\mathscr{H})$ extends to a real-linear isometry. Moreover, in 2014, Hatori and Molnár [9] generalized their result by proving that every surjective isometry between the unitary groups of two von Neumann algebras extends to a real-linear isometry. What is interesting is that, although the Mazur–Ulam type problem on unitary groups can be viewed as a localization of Tingley's problem, the method used in [9] was completely different from those studied in the context of Tingley's problem. Actually, the proof of the result in [9] mentioned above is based on C^*-algebraic techniques such as Stone's theorem and the result of Kadison [11].

However, in the case of C^*-algebras, Tingley's problem can be closely related to the Mazur–Ulam type problem on unitary groups. Indeed, recently, it was shown in [20] that Tingley's problem has an affirmative answer for the case of $X = Y = \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is finite dimensional (that is, $\mathscr{B}(\mathscr{H})$ is the algebra of all $n \times n$ complex matrices for some $n \in \mathbb{N}$). The solution is strongly based on the above mentioned result of Hatori and Molnár [8]; and C^*-algebraic methods are still effective for Tingley's problem on C^*-algebras.

The main purpose of this paper is to present, using both C^*-algebraic and Banach space geometric methods, a solution of Tingley's problem for the case of finite dimensional C^*-algebras. More precisely, it is shown that every surjective isometry between the unit spheres of two finite dimensional C^*-algebras extends to a real-linear Jordan $*$-isomorphism followed by multiplied by a fixed unitary element. Then, furthermore, we study the impact of the existence of surjective isometries between the unit spheres of two finite dimensional C^*-algebras. It turns out that if two finite dimensional C^*-algebras have isometric unit spheres, then they are $*$-isomorphic.

2. Extensions of spherical isometries

We start with the following basic result. The proof is based on Eidelheit's separation theorem [17, Theorem 2.2.26]; see, for example, [20] for the proof.

Lemma 2.1. *Let X be a Banach space. Suppose that C is a maximal convex subset of the unit sphere $S(X)$ of X. Then C is a norm exposed face of $B(X)$.*

We need the following result shown in [2, Lemma 5.1] (and [19, Lemma 3.5]).

Lemma 2.2. *Let X, Y be Banach spaces, and let $T : S(X) \to S(Y)$ be a surjective isometry. Then C is a maximal convex subset of $S(X)$ if and only if $T(C)$ is that of $S(Y)$.*

Let $\mathscr{R}$ be a von Neumann algebra. As was shown in [7, Theorem 5.3] (see also [1, Theorem 4.4]), every weak-operator closed face F of $B(\mathscr{R})$ is associated with a (unique) partial isometry $V \in \mathscr{R}$ under the equation

$$F = V + (1 - VV^*)B(\mathscr{R})(1 - V^*V) = \{A \in B(\mathscr{R}) : AV^* = VV^*\}.$$

In particular, if $\mathfrak{A}$ is a finite dimensional C^*-algebra, it can be viewed as a von Neumann algebra (by considering any faithful representation). Hence each norm closed (hence compact) face of $B(\mathfrak{A})$ has such a form.

The following is a key ingredient for our main result.

Lemma 2.3. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras with $\dim \mathfrak{A}_1 < \infty$. Suppose that T is a surjective isometry from $S(\mathfrak{A}_1)$ onto $S(\mathfrak{A}_2)$. Then $\dim \mathfrak{A}_2 < \infty$ and T is locally affine, that is, if $A, B, tA + (1-t)B \in S(\mathfrak{A}_1)$ for some $t \in (0,1)$, then $sA + (1-s)B \in S(\mathfrak{A}_1)$ for all $s \in [0,1]$ and the equation*

$$T(sA + (1-s)B) = sT(A) + (1-s)T(B)$$

holds for every $s \in [0,1]$.

Proof. Since $\dim \mathfrak{A}_1 < \infty$, its unit sphere $S(\mathfrak{A}_1)$ is norm compact; and so is its image $S(\mathfrak{A}_2)$ under the continuous mapping T. This shows that $\dim \mathfrak{A}_2 < \infty$.

Let $A, B \in S(\mathfrak{A}_1)$ be such that $tA + (1-t)B \in S(\mathfrak{A}_1)$ for some $t \in (0,1)$. Then the interval $[A, B] = \{sA + (1-s)B : s \in [0,1]\}$ is a convex subset of $S(\mathfrak{A}_1)$; and, by Zorn's lemma, it is contained in some maximal convex subset C_1 of $S(\mathfrak{A}_1)$. By Lemma 2.2, the image $T(C_1)$ of C_1 under T is also a maximal convex subset C_2 of $S(\mathfrak{A}_2)$. Since C_1 and C_2 are both closed faces by Lemma 2.1, there exist partial isometries $V_1 \in \mathfrak{A}_1$ and $V_2 \in \mathfrak{A}_2$ corresponding to C_1 and C_2, respectively.

Letting $E_j = V_j^* V_j$ and $F_j = V_j V_j^*$ yields $C_j = V_j + (1 - F_j)B(\mathfrak{A}_j)(1 - E_j)$ for $j = 1, 2$. Since the translation $T_j : A \to A - V_j$ under $-V_j$ is affine isometry on $\mathfrak{A}_j$, the face C_j is affinely isometric to the unit ball $(1 - F_j)B(\mathfrak{A}_j)(1 - E_j)$ of a Banach subspace $(1 - F_j)\mathfrak{A}_j(1 - E_j)$ of $\mathfrak{A}_j$. Now let $T_0 = T_2 T T_1^{-1}$. Then T_0 is a surjective isometry from $(1 - F_1)B(\mathfrak{A}_1)(1 - E_1)$ onto $(1 - F_2)B(\mathfrak{A}_2)(1 - E_2)$. The Mankiewicz theorem guarantees that T_0 is affine (as the restriction of the extended linear isometry), and so is $T|C_1 = T_2^{-1} T_0 T_1$. This shows that T is locally affine. $\square$

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras. We recall that a linear mapping $J : \mathfrak{A}_1 \to \mathfrak{A}_2$ is called a *Jordan $*$-homomorphism* if it satisfies $J(A^2) = J(A)^2$ and $J(A^*) = J(A)^*$ for each $a \in \mathfrak{A}_1$. If J is real-linear, it is described as a *real-linear Jordan $*$-homomorphism*. By a (real-linear) Jordan $*$-*isomorphism*, we mean a bijective (real-linear) Jordan $*$-homomorphism.

Let $J : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a Jordan $*$-isomorphism. Then

(i) $J(I) = I$;
(ii) $J(AB) = J(A)J(B)$ whenever $AB = BA$; and
(iii) $J(E)$ is a projection if and only if E is a projection.

See, for example, [13, Exercise 10.5.22] and its solution in [14]. In particular, J preserves the centers and unitary groups. Moreover, by the Kadison–Pedersen theorem [12], each element in the open unit ball of a unital C^*-algebra can be written as the mean of finitely many unitary elements in it. From this, in particular, each Jordan $*$-isomorphism is an isometry; see also [18] for some other useful facts. Consequently, if J is a Jordan $*$-isomorphism, the formula $T(A) = T(I)(PJ(A) + (I - P)J(A)^*)$ defines a real-linear isometric isomorphism.

The following is the main result in this paper, and provides an affirmative answer to Tingley's problem for surjective isometries between the unit spheres of two finite dimensional C^*-algebras. The proof is based on the preceding lemma, and the result of Hatori and Molnár [9, Corollary 3].

Theorem 2.4. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be finite dimensional C^*-algebras. Suppose that T is a surjective isometry from $S(\mathfrak{A}_1)$ onto $S(\mathfrak{A}_2)$. Then there exist a central projection $P \in \mathfrak{A}_2$ and a Jordan $*$-isomorphism $J : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that*

$$T(A) = T(I)(PJ(A) + (I - P)J(A)^*)$$

for each $A \in S(\mathfrak{A}_1)$. Consequently, the mapping T extends (uniquely) to a real-linear Jordan $*$-isomorphism followed by multiplication operator by a fixed unitary element.

Proof. We first note that the set of all extreme points $\operatorname{ext}\mathscr{B}(\mathscr{R})$ of the unit ball of a finite von Neumann algebra $\mathscr{R}$ coincides with its unitary group $\mathscr{U}$; see, for example, [16, Lemma 2]. Since finite dimensional C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be viewed as finite von Neumann algebras, one has that $\operatorname{ext}B(\mathfrak{A}_j) = \mathscr{U}_j$ for $j = 1, 2$, where $\mathscr{U}_j$ is the unitary group of $\mathfrak{A}_j$.

We next show that $T(\mathscr{U}_1) = \mathscr{U}_2$. Let $U \in \mathscr{U}_1$, and let $T(U) = 2^{-1}(A + B)$ with $A, B \in S(\mathfrak{A}_2)$. Applying Lemma 2.3 (to T^{-1}) yields

$$U = T^{-1}\left(\frac{A+B}{2}\right) = \frac{T^{-1}(A) + T^{-1}(B)}{2},$$

which implies that $U = T^{-1}(A) = T^{-1}(B)$ by since U is an extreme point, and so $T(U) = A = B$. This proves that $T(U) \in \mathscr{U}_2$; and $T(\mathscr{U}_1) \subset \mathscr{U}_2$. Interchanging the roles of T and T^{-1} in the above argument, we have the reverse inclusion. Hence it follows that $T(\mathscr{U}_1) = \mathscr{U}_2$.

Now the result of Hatori and Molnár [9, Corollary 3] applies, and there exist a central projection $P \in \mathfrak{A}_1$ and a Jordan $*$-isomorphism $J : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that

$$T(U) = T(I)(PJ(U) + (I - P)J(U)^*)(= S(U))$$

for each $U \in \mathscr{U}_1$. We note that $T(I)$ is a unitary element since $T(I) \in T(\mathscr{U}_1) = \mathscr{U}_2$. It remains to show that this equation holds for each $A \in S(\mathfrak{A}_1)$. To see this, recall that each element in the unit ball of a finite von Neumann algebra can be represented as the midpoint of two unitaries. So each $A \in S(\mathfrak{A}_1)$ is written in the form $A = 2^{-1}(U + V)$, where $U, V \in \mathscr{U}_1$. Then it follows that

$$T(A) = \frac{T(U) + T(V)}{2} = \frac{S(U) + S(V)}{2} = S(A)$$

since T is locally affine by Lemma 2.3. The proof is complete. □

We remark, by Theorem 2.4 (and Lemma 2.3), that if $\dim \mathfrak{A}_1 < \infty$ and there exists an isometry from $S(\mathfrak{A}_1)$ onto $S(\mathfrak{A}_2)$, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic as Jordan $*$-algebras (under a Jordan $*$-isomorphism J derived in that theorem). Since the algebras under consideration are finite dimensional, in fact, they are $*$-isomorphic. Thus we obtain the following consequence of the main theorem.

Corollary 2.5. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be finite dimensional C^*-algebras. Then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $*$-isomorphic if and only if $S(\mathfrak{A}_1)$ and $S(\mathfrak{A}_2)$ are isometric as metric spaces.*

Remark 2.6. It should be noted that Corollary 2.5 does not hold for general von Neumann algebras. Indeed, we have a von Neumann factor $\mathscr{M}$ that is not $*$-anti-isomorphic to itself (Connes [3]). Suppose that ρ is a faithful normal semi-finite weight on $\mathscr{M}$. Then, by Tomita's theorem (extended to weights; see, e.g., [13, Theorem 9.2.37]), there exist a faithful normal representation $\pi : \mathscr{M} \to \mathscr{B}(\mathscr{H}_\rho)$ and a conjugate-linear isometry J acting on $\mathscr{H}_\rho$ satisfying $J^2 = I$ (hence $J = J^{-1} = J^*$) and $J\pi(\mathscr{M})J = \pi(\mathscr{M})'$. From this, the mapping $A \to JA^*J : \pi(\mathscr{M}) \to \pi(\mathscr{M})'$ is a $*$-anti-isomorphism. Let $\mathscr{M}_a$ be the von Neumann algebra $\pi(\mathscr{M})'$. Then $\mathscr{M}$ and $\mathscr{M}_a$ are $*$-anti-isomorphic. Let $\mathscr{R}_1 = \mathscr{M} \oplus \mathscr{M}$ and $\mathscr{R}_2 = \mathscr{M} \oplus \mathscr{M}_a$. If φ is a $*$-anti-isomorphism from $\mathscr{M}$ onto $\mathscr{M}_a$. Then the mapping $A \oplus B \to A \oplus \varphi(B) : \mathscr{R}_1 \to \mathscr{R}_2$ is a Jordan $*$-isomorphism. In particular, $S(\mathscr{R}_1)$ and $S(\mathscr{R}_2)$ are isometric as metric spaces. However, two von Neumann algebras $\mathscr{R}_1$ and $\mathscr{R}_2$ can not be $*$-isomorphic nor $*$-anti-isomorphic to each other.

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